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Skolem properties, value-functions, and divisorial ideals

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Abstract

Let D be the ring of integers of a number field K. It is well known that the ring $\mathrm{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}$ of integer-valued polynomials on D is a Prüfer domain. Here we study the divisorial ideals of $\mathrm{Int}(D)$ and prove in particular that $\mathrm{Int}(D)$ has no divisorial prime ideal.

We begin with the local case. We show that, if V is a rank-one discrete valuation domain with finite residue field, then the unitary ideals of $\mathrm{Int}(V)$ (that is, the ideals containing nonzero constants) are entirely determined by their values on the completion of V. This improves on the Skolem properties which only deal with finitely generated ideals. We then globalize and consider a Dedekind domain D with finite residue fields. We show that a prime ideal of $\mathrm{Int}(D)$ is invertible if and only if it is divisorial, and also, in the case where the characteristic of D is 0, if and only if it is an upper to zero which is maximal.

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0. Introduction

If D is a domain, with quotient field K, we consider the ring $\mathrm{Int}(D)$ of integer-valued polynomials on D:

$$\mathrm{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}.$$

For each ideal $\mathfrak{A}$ of $\mathrm{Int}(D)$, and each $a \in D$, $\mathfrak{A}(a) = \{g(a) \mid g \in \mathfrak{A}\}$ is clearly an ideal of $\mathrm{Int}(D)$, quite naturally called the *ideal of values* of $\mathfrak{A}$ at a. The various *Skolem properties* of $\mathrm{Int}(D)$ give a measure as to what extent an ideal is characterized 1

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its ideal of values [1–3, 8, 9, 11, 17]. We say that Int(D) has the *Skolem property* if each finitely generated ideal $\mathfrak{A}$ of Int(D) such that $\mathfrak{A}(a)=D$ for all $a\in D$ is, in fact, equal to Int(D); we say that Int(D) has the *strong Skolem property* if, whenever two finitely generated ideals $\mathfrak{A}$ and $\mathfrak{B}$ are such that $\mathfrak{A}(a)=\mathfrak{B}(a)$ for all $a\in D$, then $\mathfrak{A}=\mathfrak{B}$. For instance, Int($\mathbb{Z}$) has the Skolem property [20] and even the strong Skolem property [3]; however, we emphasize that this property is restricted to finitely generated ideals.

In this paper we first deal with the case where $D=V$ is a rank-one discrete valuation domain with finite residue field. Since V is a local ring, we cannot expect Int(V) to have the Skolem property, let alone the strong Skolem property, indeed the principal ideal $\mathfrak{A}$ generated by a unit-valued polynomial (for instance, $1+tX$ where t is in the maximal ideal) is such that $\mathfrak{A}(a)=V$ for each $a\in V$. However we do have the Skolem properties restricted to the *unitary ideals*, that is, the ideals containing nonzero constants. These are the *almost Skolem* and *almost strong Skolem* properties (that McQuillan called the Hilbert and strong Hilbert properties [18]).

Letting $\mathfrak{m}$ be the maximal ideal of V, an integer-valued polynomial can be seen as a uniformly continuous function in the $\mathfrak{m}$-adic topology, hence as a continuous function from the completion $\widehat{V}$ of V into itself. We extend the notion of ideal of values and associate to each ideal $\mathfrak{A}$ of Int(D) a *value-function* $\Psi_{\mathfrak{A}}$ (on $\widehat{V}$). We then show that the unitary ideals are entirely characterized by their value-functions. We emphasize that there is no longer any restriction to the finitely generated ideals (which simply turn out to be those ideals whose value-functions are locally constant).

Extending these notions to fractional ideals, we then compare the value-functions of an ideal $\mathfrak{A}$ and its inverse $\mathfrak{A}^{-1}$. We give conditions for an ideal $\mathfrak{A}$ to be divisorial, that is, $\mathfrak{A}=(\mathfrak{A}^{-1})^{-1}$, and then easily exhibit an ideal which is divisorial but not finitely generated. (Recall that Int(V) is a Prüfer domain, that is, every finitely generated ideal is invertible [10, Proposition 2.3].)

Finally, we globalize and consider the case where D is a Dedekind domain with finite residue fields. We characterize the unitary ideals by the collection of their $\mathfrak{m}$-adic value-functions, at every maximal ideal $\mathfrak{m}$ of D. Then we show that a prime ideal of Int(D) is invertible if and only if it is divisorial, and also, in the case where the characteristic of D is 0, if and only if it is an upper to zero which is maximal. It follows that if D is the ring of integers of an algebraic number field, in particular if $D=\mathbb{Z}$ (the ring of integers), then no prime ideal of Int(D) is divisorial.[1]

[1] This paper was under preparation while the first two authors were finishing their book on integer-valued polynomials. This book has now appeared [5]. Some results, dealing with the local case, were included in the book; for consistency they are stated and proved in this paper. Note that our point of view here is somewhat different: for simplicity, we restrict ourselves to a discrete valuation domain V; moreover, a value-function is considered as a function on the completion $\widehat{V}$ (whereas, in the book, it is considered as a function on V). Note also that the results derived here by globalization for a Dedekind domain are completely new.

1. Value-functions of unitary ideals and Skolem properties

We let V be a rank-one discrete valuation domain with finite residue field, $\mathfrak{m}$ be its maximal ideal, v be the corresponding valuation, and t be a generator of the maximal ideal. A polynomial can be viewed as a uniformly continuous function (from V to K) in the $\mathfrak{m}$-adic topology. We also consider the completion $\widehat{V}$ of V, and we denote by $\widehat{\mathfrak{m}}$ its maximal ideal. We continue to denote by v the extension of the valuation to the completion.

Let $\mathfrak{A}$ be an ideal of $\mathrm{Int}(V)$. For $a \in V$, if the ideal of values $\mathfrak{A}(a)$ is a nonzero ideal, it is of the form $\mathfrak{m}^n$, where $n = \inf\{v(f(a)) \mid f \in \mathfrak{A}\}$. We thus define a function from V to $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$: to each $a \in V$ corresponds the integer n, if $\mathfrak{A}(a) \neq (0)$, and ∞ otherwise ($v(0) = \infty$). In fact, we can extend this function to $\widehat{V}$, considering an integer-valued polynomial as a continuous function from $\widehat{V}$ to $\widehat{V}$ (in the $\mathfrak{m}$-adic topology). For each ideal $\mathfrak{A}$ of $\mathrm{Int}(V)$, we finally obtain a function

$$\Psi_{\mathfrak{A}} : \widehat{V} \to \overline{\mathbb{N}}$$

such that

$$\Psi_{\mathfrak{A}}(x) = \inf\{v(f(x)) \mid f \in \mathfrak{A}\}$$

for $x \in \widehat{V}$. We say that $\Psi_{\mathfrak{A}}$ is the *value-function* of the ideal $\mathfrak{A}$.

If f is an integer-valued polynomial, we denote by Ψ_f the value-function of the ideal generated by f, that is, for each $x \in \widehat{V}$:

$$\Psi_f(x) = v(f(x)).$$

For each ideal $\mathfrak{A}$ of $\mathrm{Int}(V)$, we then have

$$\Psi_{\mathfrak{A}} = \inf\{\Psi_f \mid f \in \mathfrak{A}\}.$$

Let us give an example. Recall that the unitary prime ideals of $\mathrm{Int}(V)$, that is, the prime ideals above $\mathfrak{m}$, are in one-to-one correspondence with the elements of $\widehat{V}$ [7, Proposition 5.4]: to each $\alpha \in \widehat{V}$ corresponds the maximal ideal

$$\mathfrak{M}_\alpha = \{f \in \mathrm{Int}(V) \mid f(\alpha) \in \widehat{\mathfrak{m}}\}.$$

The value-function of $\mathfrak{M}_\alpha$ is the function such that

$$\Psi_{\mathfrak{M}_\alpha}(x) = \begin{cases} 1, & \text{if } x = \alpha, \\ 0, & \text{if } x \neq \alpha. \end{cases}$$

We next prove that the value-function $\Psi_{\mathfrak{A}}$ of an ideal $\mathfrak{A}$ must satisfy a property of continuity.

Lemma 1.1. *For each ideal $\mathfrak{A}$ of $\mathrm{Int}(V)$, the value-function $\Psi_{\mathfrak{A}} : \widehat{V} \to \overline{\mathbb{N}}$ is upper semicontinuous, where $\overline{\mathbb{N}}$ is endowed with the discrete topology. That is, for each $x \in \widehat{V}$, there exists a neighborhood U of x such that $\Psi_{\mathfrak{A}}(y) \leq \Psi_{\mathfrak{A}}(x)$ for each $y \in U$.*

Proof. Let x be in $\widehat{V}$. If $\Psi_{\mathfrak{A}}(x)=\infty$, the assertion is obvious. Hence we may assume that $\Psi_{\mathfrak{A}}(x)\neq\infty$: there is $g\in\mathfrak{A}$ such that $v(g(x))=\Psi_{\mathfrak{A}}(x)$. Since g is a continuous function, there is a neighborhood U of x such that, for each $y\in U$, one has $v(g(y))=v(g(x))$, and in particular, $\Psi_{\mathfrak{A}}(y)\leq v(g(y))=\Psi_{\mathfrak{A}}(x)$. □

If an ideal $\mathfrak{A}$ is unitary, that is, contains nonzero constants, its ideals of values are nonzero ideals: they contain the constants which lie in $\mathfrak{A}$. Hence $\Psi_{\mathfrak{A}}$ is a function from $\widehat{V}$ to $\mathbb{N}$. Since $\Psi_{\mathfrak{A}}$ is upper semicontinuous and $\widehat{V}$ is compact, let us note also that $\Psi_{\mathfrak{A}}$ is bounded above. Using the fact that $\widehat{V}$ is compact and the Stone–Weierstrass theorem [5, Theorem III.3.4] (every continuous function from $\widehat{V}$ to $\widehat{V}$ can be uniformly approximated by an integer-valued polynomial), we can prove the following converse of Lemma 1.1.

Lemma 1.2. *Every upper semicontinuous function $\Psi:\widehat{V}\to\mathbb{N}$ is the value function of a unitary ideal $\mathfrak{A}$ of* Int(V).

Proof. Let $\mathfrak{A}=\{g\in\mathrm{Int}(V)\mid \Psi_g\geq\Psi\}$. Clearly $\mathfrak{A}$ is an ideal. We show that it is the ideal we are looking for. As Ψ is upper semicontinuous and $\widehat{V}$ is compact, Ψ is bounded above by an integer M. Hence, t^M belongs to $\mathfrak{A}$ and $\mathfrak{A}$ is unitary. By definition $\Psi_{\mathfrak{A}}=\inf\{\Psi_g\mid g\in\mathfrak{A}\}\geq\Psi$. It remains to show that $\Psi_{\mathfrak{A}}\leq\Psi$. Fix $x\in\widehat{V}$, and let $n=\Psi(x)$. There exists a clopen neighborhood U of x such that $\Psi(y)\leq n$ for each $y\in U$. By the Stone–Weierstrass theorem there exists a polynomial $f\in\mathrm{Int}(V)$ such that $v(f(y))=n$ for each $y\in U$, and $v(f(y))\geq M$ for each $y\notin U$. Then $\Psi_f\geq\Psi$, and hence, $f\in\mathfrak{A}$. Consequently, $\Psi_{\mathfrak{A}}(x)\leq\Psi_f(x)=n=\Psi(x)$. □

The next theorem shows that the unitary ideals, are entirely determined by their value-functions [5, Theorem VII.3.7]. Again, we use compactness of $\widehat{V}$ and the Stone–Weierstrass theorem.

Theorem 1.3. *Let $\mathfrak{A}$ be a unitary ideal of* Int(V), *and $f\in\mathrm{Int}(V)$. Then $f\in\mathfrak{A}$ if and only if $\Psi_f\geq\Psi_{\mathfrak{A}}$.*

Proof. By definition, if $f\in\mathfrak{A}$, then $\Psi_f\geq\Psi_{\mathfrak{A}}$. We must prove the converse. Since $\mathfrak{A}$ is unitary, it contains a nonzero constant a, we let $k=v(a)$. Suppose that $\Psi_f\geq\Psi_{\mathfrak{A}}$. Then, for $x\in\widehat{V}$, $f(x)$ belongs to the ideal $\mathfrak{A}(x)\widehat{V}$ (generated in $\widehat{V}$ by the ideal of values $\mathfrak{A}(x)$). Since $\mathfrak{A}(x)$ is dense in $\mathfrak{A}(x)\widehat{V}$, there is in particular a polynomial $g_x\in\mathfrak{A}$ such that $v(f(x)-g_x(x))\geq k$. Since f and g_x are continuous, there is a clopen neighborhood U_x of x such that $v(f(z)-g_x(z))\geq k$, for each $z\in U_x$. Since $\widehat{V}$ is compact, it can be covered by finitely many such clopen sets, say $U_1,\ldots,U_s$, and there are corresponding elements $g_1,\ldots,g_s$ of $\mathfrak{A}$ such that $v(f(z)-g_i(z))\geq k$ for $z\in U_i$. These subsets being clopen sets, we can require them to have pairwise empty intersections. From the Stone–Weierstrass theorem the characteristic function of each clopen set U_i can be uniformly approximated modulo $\mathfrak{m}^k$ by an integer-valued polynomial φ_i. Let $z\in\widehat{V}$. Then z belongs to one and only one of these clopen sets, say U_j; we have $\varphi_j(z)=1+\alpha_j$, and $\varphi_i(z)=\alpha_i$, for $i\neq j$, where $v(\alpha_i)\geq k$ for each i. Letting $g=\sum_i g_i\varphi_i$, then $g\in\mathfrak{A}$ and,

$$f(z)-g(z)=f(z)-\varphi_j(z)g_j(z)-\sum_{i\neq j}\varphi_i(z)g_i(z)=f(z)-g_j(z)-\sum_i \alpha_i g_i(z).$$

Thus, for each $z\in\widehat{V}$, we have $v(f(z)-g(z))\geq k$. Hence $h=a^{-1}(f-g)\in\mathrm{Int}(V)$. Finally $f=g+ah$ belongs to $\mathfrak{A}$. □

Corollary 1.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two ideals of* Int(V).

(i) *Suppose that $\mathfrak{B}$ is unitary. Then $\mathfrak{A}\subseteq\mathfrak{B}$ if and only if $\Psi_{\mathfrak{A}}\geq\Psi_{\mathfrak{B}}$.*

(ii) *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are unitary. Then $\mathfrak{A}=\mathfrak{B}$ if and only if $\Psi_{\mathfrak{A}}=\Psi_{\mathfrak{B}}$.*

The last assertion expresses a Skolem property with respect to the unitary (but not necessarily finitely generated) ideals of Int(V).

From Lemma 1.2 and Corollary 1.4, we immediately derive the following.

Proposition 1.5. *Every upper semicontinuous function $\Psi:\widehat{V}\to\mathbb{N}$ is the value function of one and only one unitary ideal $\mathfrak{A}$ of* Int(V).

Finally, we characterize the unitary finitely generated ideals. If the unitary ideal $\mathfrak{A}$ is generated by $g_1,\ldots,g_k$, then, for each $x\in\widehat{V}$, there exists a neighborhood U of x such that $v(g_i(y))=v(g_i(x))$ for each $y\in U$ and each $i\in\{1,\ldots,k\}$. Hence, $\Psi_{\mathfrak{A}}(y)=\Psi_{\mathfrak{A}}(x)$ for each $y\in U$. Therefore the value-function $\Psi_{\mathfrak{A}}$ is continuous, that is to say, locally constant. In fact, $\Psi_{\mathfrak{A}}$ takes only finitely many distinct values since $\widehat{V}$ is compact. Conversely, we have the following:

Proposition 1.6. *Every locally constant function $\Psi:\widehat{V}\to\mathbb{N}$ is the value-function of one and only one finitely generated unitary ideal $\mathfrak{A}$ of* Int(V).

Proof. Let $\mathfrak{A}=\{g\in\mathrm{Int}(V)\mid\Psi_g\geq\Psi\}$, as in Lemma 1.2. It follows from the previous results that $\mathfrak{A}$ is the unique unitary ideal with value-function Ψ; it remains to show that it is finitely generated. The topological space $\widehat{V}$ is covered by finitely many disjoint open sets $U_1,\ldots,U_r$ and there are integers $n_1,\ldots,n_r$ such that $\Psi(a)=n_i$, for $a\in U_i$. From the Stone–Weierstrass theorem there exists a polynomial $f\in\mathrm{Int}(V)$ such that $v(f(x))=n_i$ for $x\in U_i$, in other words, such that $\Psi_f=\Psi$. It is now straightforward to show that $\mathfrak{A}=(a,f)$, where a is an arbitrarily chosen nonzero constant in $\mathfrak{A}$. □

From Corollary 1.4 and Proposition 1.6, we immediately derive the following.

Corollary 1.7. *The value-function $\Psi_{\mathfrak{A}}$ of a unitary ideal $\mathfrak{A}$ is locally constant if and only if $\mathfrak{A}$ is finitely generated.*

Remark 1.8. (1) Following Gilmer and Smith [14, Theorem 2.8], we could say in the case of Int($\mathbb{Z}$) that the value-function, restricted to $\mathbb{Z}$, of a unitary finitely generated ideal is periodic.

(2) In general, the nonunitary ideals are not characterized by their ideals of values. Consider for instance the polynomial $f = 1 + tX$. For each n the principal ideal (f^n) is such that its ideal of values is everywhere equal to D. Yet these ideals are clearly distinct.

(3) In general, the unitary ideals are not characterized by their ideals of values on V only (that is, by the restriction of their value-function to V). For instance if $\mathfrak{M}_\alpha$ is a maximal ideal corresponding to $\alpha \notin V$, then $\Psi_{\mathfrak{A}}(a) = 0$ for each $a \in V$, while $\mathfrak{M}_\alpha \neq \mathrm{Int}(V)$. Note also that, in this case, $\Psi_{\mathfrak{M}_\alpha}$ restricted to V is locally constant (in fact, constant) while $\mathfrak{M}_\alpha$ is not finitely generated.

(4) If we restrict ourselves to finitely generated ideals, it is known that $\mathfrak{A}$ and $\mathfrak{B}$ are equal if and only if $\Psi_{\mathfrak{A}}(a) = \Psi_{\mathfrak{B}}(a)$ for each $a \in V$ (this is the almost strong Skolem or strong Hilbert property [18, Lemma 2.6]). This can easily be derived from the previous results. Indeed, suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are finitely generated and that $\Psi_{\mathfrak{A}}(a) = \Psi_{\mathfrak{B}}(a)$ for each $a \in V$. Then $\Psi_{\mathfrak{A}} = \Psi_{\mathfrak{B}}$, since $\Psi_{\mathfrak{A}}$ and $\Psi_{\mathfrak{B}}$ are locally constant and V is dense in $\widehat{V}$. Hence $\mathfrak{A} = \mathfrak{B}$.

(5) Note that our proof of Proposition 1.6 actually provides a new demonstration of the fact that a finitely generated unitary ideal of $\mathrm{Int}(V)$ can be generated by two elements, one of them being an arbitrarily chosen nonzero constant [10, Proposition 3.5].

(6) Assuming that D is a one-dimensional local Noetherian domain, with finite residue field, the Stone–Weierstrass theorem applies if (and only if) D is also analytically irreducible [6, Theorem p. 53]. We could generalize the results of this section to this situation. In particular the unitary ideals are again characterized by their ideals of values: $\mathfrak{A} = \mathfrak{B}$ if and only if $\mathfrak{A}(x)\widehat{D} = \mathfrak{B}(x)\widehat{D}$ for each $x \in \widehat{D}$ [5, Theorem VII.3.7].

2. Value-functions of fractional ideals

Since we wish to study invertible and divisorial ideals, it is natural to extend the previous notions to fractional ideals. We define the value-function Ψ_φ of a rational function φ by $\Psi_\varphi(x) = v(\varphi(x))$. If x is a root of φ, then $\Psi_\varphi(x) = \infty$ and if x is a pole, then $\Psi_\varphi(x) = -\infty$. Then we define the value-function of a fractional ideal $\mathfrak{A}$ by

$$\Psi_{\mathfrak{A}}(x) = \inf\{\Psi_\varphi(x) \mid \varphi \in \mathfrak{A}\} = \inf\{v(\varphi(x)) \mid \varphi \in \mathfrak{A}\}.$$

It is a function from $\widehat{V}$ to $\mathbb{Z} \cup \{-\infty, \infty\}$. In fact, if $\mathfrak{A}$ is a nonzero fractional ideal, $\Psi_{\mathfrak{A}}(x)$ is an integer for every $x \in \widehat{V}$ except at most finitely many. Indeed $\mathfrak{A}$ contains a nonzero polynomial f, and $\Psi_{\mathfrak{A}}(x) \neq \infty$ whenever x is not a root of f; also, since $\mathfrak{A}$ is a fractional ideal, there is a nonzero polynomial g such that $g\mathfrak{A}$ is an integral ideal, and $\Psi_{\mathfrak{A}}(x) \neq -\infty$, whenever x is not a root of g.

We first list some elementary properties of the value-functions. The first one is immediate:

Lemma 2.1. *Let $\mathfrak{A}, \mathfrak{B}$ be two fractional ideals of* Int(V) *and $x \in \widehat{V}$ such that $\Psi_{\mathfrak{A}}(x)$ and $\Psi_{\mathfrak{B}}(x)$ are integers. Then $\Psi_{\mathfrak{A}\mathfrak{B}}(x) = \Psi_{\mathfrak{A}}(x) + \Psi_{\mathfrak{B}}(x)$. In particular, if $\Psi_{\mathfrak{A}}$ and $\Psi_{\mathfrak{B}}$ take their values in $\mathbb{Z}$, then $\Psi_{\mathfrak{A}\mathfrak{B}} = \Psi_{\mathfrak{A}} + \Psi_{\mathfrak{B}}$.*

Corresponding to Lemma 1.1, we also have the following.

Lemma 2.2. *Let $\mathfrak{A}$ be a fractional ideal. The value-function $\Psi_{\mathfrak{A}}$ is upper semicontinuous at each point x such that $\Psi_{\mathfrak{A}}(x) \neq -\infty$.*

Proof. The result is clear if $\Psi_{\mathfrak{A}}(x) = \infty$. Otherwise there is a rational function $\varphi = f/g$ in $\mathfrak{A}$ such that x is not a root of f and g and $\Psi_{\varphi}(x) = \Psi_{\mathfrak{A}}(x)$. Thus $\Psi_{\varphi} = \Psi_f - \Psi_g$ is continuous (that is, constant) in a neighborhood U of x. Since $\Psi_{\mathfrak{A}} \leq \Psi_{\varphi}$, $\Psi_{\mathfrak{A}}$ is upper semicontinuous at x (that is, $\Psi_{\mathfrak{A}}(y) \leq \Psi_{\mathfrak{A}}(x)$ for each $y \in U$). □

We may generalize the results of the previous section to a class of fractional ideals. Let us first give a definition:

Definition 2.3. We say that a fractional ideal $\mathfrak{A}$ of Int(V) is *almost-unitary* if there is a nonzero constant $a \in K$ such that $a\mathfrak{A}$ is unitary, that is, $a\mathfrak{A}$ is contained in Int(V) and contains nonzero constants.

Of course the nonzero element a can be taken in V. Moreover, the value-function of an almost-unitary ideal is bounded below. We leave to the reader the verifications of the following assertions:

- Let $\mathfrak{A}$ be an almost-unitary ideal and $f \in K[X]$. Then $\Psi_f \geq \Psi_{\mathfrak{A}}$ if and only if $f \in \mathfrak{A}$.
- Let $\mathfrak{A}$ and $\mathfrak{B}$ be two almost-unitary ideals. Then $\mathfrak{A} = \mathfrak{B}$ if and only if $\Psi_{\mathfrak{A}} = \Psi_{\mathfrak{B}}$.
- An upper semicontinuous function from $\widehat{V}$ to $\mathbb{Z}$ which is bounded below is the value-function of one and only one almost-unitary ideal.
- A locally constant function (from $\widehat{V}$ to $\mathbb{Z}$) is the value-function of one and only one finitely generated almost-unitary ideal.
- The value-function of an almost-unitary ideal $\mathfrak{A}$ is locally constant if and only if $\mathfrak{A}$ is finitely generated.

The value-function of an integral ideal of Int(V) is positive, hence it is bounded below. The value-function of a unitary ideal $\mathfrak{A}$ is bounded above (by the valuation of every nonzero constant $d \in \mathfrak{A}$). More generally, the value-function of an almost-unitary ideal is bounded. However, the converse does not hold. For instance, the principal ideal generated by the polynomial $1 + tX$ is not almost-unitary, yet its value-function is everywhere null. In fact, we may characterize the bounded-valued ideals as follows.

Proposition 2.4. *The value-function of a fractional ideal $\mathfrak{A}$ is bounded if and only if there is a rational function φ with no root and no pole in $\widehat{V}$ and a unitary ideal $\mathfrak{B}$ such that $\mathfrak{A} = \varphi\mathfrak{B}$.*

Proof. If $\mathfrak{A} = \varphi\mathfrak{B}$, where $\mathfrak{B}$ is unitary and φ has no root and no pole, then $\Psi_{\mathfrak{A}}$ is bounded since $\Psi_{\mathfrak{A}} = \Psi_{\varphi} + \Psi_{\mathfrak{B}}$ (Lemma 2.1) (and Ψ_{φ} is, in fact, locally constant).

Conversely, suppose that $\Psi_{\mathfrak{A}}$ is bounded. Write $\mathfrak{A}K[X] = \psi K[X]$. We claim that ψ has no root and no pole in $\widehat{V}$. Indeed, a root of ψ would be a common root of the elements of $\mathfrak{A}$, and this would imply $\Psi_{\mathfrak{A}}(x) = \infty$. On the other hand, there is a nonzero constant a such that $a\psi \in \mathfrak{A}$; if x were a pole of ψ, we would have $\Psi_{\mathfrak{A}}(x) \leq \Psi_{a\psi}(x) = -\infty$. Then Ψ_{ψ} is bounded. The ideal $\psi^{-1}\mathfrak{A}$ contains some nonzero constants, it is contained in $K[X]$, and its value-function is bounded. In particular it is bounded below and, for some integer n, the value-function of the ideal $\mathfrak{B} = t^n\psi^{-1}\mathfrak{A}$ is positive. Therefore $\mathfrak{B}$ is a unitary ideal of $\text{Int}(V)$ and $\mathfrak{A} = \varphi\mathfrak{B}$, where $\varphi = t^{-n}\psi$ has no root and no pole. $\square$

If the value-function of a fractional ideal never takes the value $+\infty$, it follows from Lemma 2.2 and compactness that it is bounded above. However, the next example shows that the value-function may fail to be bounded below even though it never takes the value $-\infty$.

Example 2.5. For each integer n, consider the clopen set $U_n = t^n + \widehat{\mathfrak{m}}^{n+2}$. From the Stone–Weierstrass theorem, there is a polynomial f_n such that

$$v(f_n(x)) = \begin{cases} -n, & \text{if } x \in U_n, \\ 0, & \text{if } x \notin U_n. \end{cases}$$

Let $\mathfrak{A}$ be the ideal generated by the polynomials $\{f_n\}_{n\in\mathbb{N}}$. The ideal $\mathfrak{A}$ is fractional: for each n, Xf_n is an integer-valued polynomial. The function $\Psi_{\mathfrak{A}}$ takes its values in $\mathbb{Z}$ ($\mathfrak{A} \subseteq K[X]$), yet it is not bounded below ($\Psi_{\mathfrak{A}}(t^n) \leq \Psi_{f_n}(t^n) = -n$).

3. Divisorial ideals

We now consider the inverse $\mathfrak{A}^{-1} = \{\varphi \in K(X) \mid \varphi\mathfrak{A} \subseteq \text{Int}(V)\}$ of a nonzero ideal $\mathfrak{A}$. If $\varphi \in \mathfrak{A}^{-1}$ and $\psi \in \mathfrak{A}$, then $\varphi\psi$ is an integer-valued polynomial. Hence, for each $x \in \widehat{V}$, we have $v(\varphi(x)\psi(x)) \geq 0$. In other words, we have the following:

Lemma 3.1. *If $\mathfrak{A}$ is a nonzero fractional ideal of* $\text{Int}(V)$, *then* $\Psi_{\mathfrak{A}^{-1}} \geq -\Psi_{\mathfrak{A}}$.

We shall see below that the inequality may be strict, even for a divisorial ideal (Example 3.8). However, if $\mathfrak{A}$ is invertible we have the following:

Proposition 3.2. *If $\mathfrak{A}$ is an invertible ideal of* $\text{Int}(V)$, *then* $\Psi_{\mathfrak{A}^{-1}} = -\Psi_{\mathfrak{A}}$.

Proof. Since $\mathfrak{A}\mathfrak{A}^{-1} = \text{Int}(V)$, there are elements $\varphi_1, \ldots, \varphi_r$ in $\mathfrak{A}$, and $\psi_1, \ldots, \psi_r$ in $\mathfrak{A}^{-1}$, such that $\varphi_1\psi_1 + \cdots + \varphi_r\psi_r = 1$. Suppose that $x \in \widehat{V}$, is not a pole of any of these functions. Then

$$\varphi_1(x)\psi_1(x) + \cdots + \varphi_r(x)\psi_r(x) = 1.$$

There is some index i such that $\varphi_i(x)\psi_i(x)$ is a unit of $\widehat{V}$, hence

$$\Psi_{\mathfrak{A}^{-1}}(x) \leq v(\psi_i(x)) = -v(\varphi_i(x)) \leq -\Psi_{\mathfrak{A}}(x).$$

We may conclude that $\Psi_{\mathfrak{A}^{-1}}(x) = -\Psi_{\mathfrak{A}}(x)$, from the previous lemma. If now x is a pole of some φ_i, then $\Psi_{\mathfrak{A}}(x) = -\infty$ and it follows again from the previous lemma that $\Psi_{\mathfrak{A}^{-1}}(x) = \infty$ (similarly, if x is a pole of some ψ_i, then $\Psi_{\mathfrak{A}^{-1}}(x) = -\infty$ and $\Psi_{\mathfrak{A}}(x) = \infty$). □

Remark 3.3. It is known that Int(V) is a Prüfer domain [10, Proposition 2.3]. This fact can easily be derived from the results of the first section: let $\mathfrak{A}$ be a nonzero finitely generated ideal of Int(V). The value-function $\Psi_{\mathfrak{A}}$ is locally constant, that is, continuous, and from the Stone–Weierstrass theorem there is a polynomial $g \in K[X]$ such that $\Psi_g = -\Psi_{\mathfrak{A}}$. If $f \in \mathfrak{A}$, then $\Psi_g + \Psi_f \geq 0$, hence fg is an integer-valued polynomial. In particular, $g \in \mathfrak{A}^{-1}$. Let $\mathfrak{I} = \mathfrak{A}\mathfrak{A}^{-1}$. This is an ideal of Int(V) and $\Psi_{\mathfrak{I}} = 0$ (since $\Psi_{\mathfrak{I}} \leq \Psi_{fg}$). From Theorem 1.3, we may conclude that $\mathfrak{I} = \text{Int}(V)$, provided we establish that $\mathfrak{I}$ is unitary. Write $\mathfrak{A}K[X] = hK[X]$, where h is a polynomial (that we may suppose to be in $\mathfrak{A}$). Then $h^{-1}\mathfrak{A} \subseteq K[X]$. Since $\mathfrak{A}$ is finitely generated, there is a nonzero constant $a \in V$ such that $ah^{-1}\mathfrak{A} \subseteq \text{Int}(V)$. Hence $ah^{-1} \in \mathfrak{A}^{-1}$. Finally $a = hah^{-1}$ is a nonzero constant in $\mathfrak{I} = \mathfrak{A}\mathfrak{A}^{-1}$.

The converse of Proposition 3.2 is false. Our next result shows that the equation $\Psi_{\mathfrak{A}^{-1}} = -\Psi_{\mathfrak{A}}$ and boundedness together imply divisoriality of $\mathfrak{A}$; Example 3.7 below shows the necessity of the boundedness condition.

Proposition 3.4. *Let $\mathfrak{A}$ be a fractional ideal of* Int(V) *such that $\Psi_{\mathfrak{A}}$ is bounded. If $\Psi_{\mathfrak{A}^{-1}} = -\Psi_{\mathfrak{A}}$, then $\mathfrak{A}$ is divisorial.*

Proof. Since $\Psi_{\mathfrak{A}}$ is bounded, write $\mathfrak{A} = \varphi\mathfrak{B}$, where $\mathfrak{B}$ is unitary and φ has no pole and no root (Proposition 2.4). Denote by $\mathfrak{B}_v$ the divisorial closure of $\mathfrak{B}$, that is, $\mathfrak{B}_v = (\mathfrak{B}^{-1})^{-1}$. Since $\mathfrak{B}$ is integral, $\mathfrak{B}_v$ is contained in Int(V) and a fortiori in $K[X]$. Assuming that $\Psi_{\mathfrak{A}^{-1}} = -\Psi_{\mathfrak{A}}$, then $\Psi_{\mathfrak{B}^{-1}} = -\Psi_{\mathfrak{B}}$, and $\Psi_{\mathfrak{B}_v} \geq -\Psi_{\mathfrak{B}^{-1}} = \Psi_{\mathfrak{B}}$ (Lemma 3.1). Therefore $\mathfrak{B}_v \subseteq \mathfrak{B}$ (Corollary 1.4), hence $\mathfrak{B}_v = \mathfrak{B}$ since the reverse containment always holds. It follows that $\mathfrak{B}$ is divisorial, and so is $\mathfrak{A} = \varphi\mathfrak{B}$. □

Let us now turn to the prime ideals of Int(V). Recall that the uppers to zero are the primes of the form $\mathfrak{P}_q = qK[X] \cap \text{Int}(V)$, where q is an irreducible polynomial of $K[X]$, while the unitary prime ideals are the ideals $\mathfrak{M}_\alpha$. Moreover, $\mathfrak{P}_q \subseteq \mathfrak{M}_\alpha$ if and only if $q(\alpha) = 0$ [7]. Recall also that the uppers to zero which are maximal are invertible (in fact, their classes generate the Picard group of Int(V) [10, Proposition 4.5]); all other primes, that is, the $\mathfrak{P}_q$ which are not maximal and the $\mathfrak{M}_\alpha$, are not divisorial. More precisely we can derive the following from [5, Theorem VIII.5.15].

Proposition 3.5. *Let $\mathfrak{P}$ be a prime ideal of* Int(V). *The following assertions are equivalent*:

(i) *the ideal $\mathfrak{P}$ is invertible,*

(ii) *the ideal $\mathfrak{P}$ is divisorial,*

(iii) *the ideal $\mathfrak{P}$ is an upper to zero which is maximal.*

In fact, if $\mathfrak{P}$ is a nonzero prime ideal which is not invertible, then $\mathfrak{P}^{-1} = \mathrm{Int}(V)$.

Remark 3.6. In [5, Theorem VIII.5.15], the previous proposition was more generally established for Int(D), with D a local one-dimensional unibranched domain. In the present situation, Int(V) is a two-dimensional Prüfer domain [10, Proposition 2.3]. We could then note that for a Prüfer domain R,

(a) a maximal ideal $\mathfrak{M}$ of R is either invertible or such that $\mathfrak{M}^{-1}=R$ [12, Corollary 3.1.3],

(b) if the dimension of R is at most two, then a prime ideal $\mathfrak{P}$ of R is either divisorial or such that $\mathfrak{P}^{-1}=R$ [12, Theorem 4.1.22].

Most of the conclusions of the previous proposition could then be derived from these results (note also that it is very easy to prove that the unitary maximal ideals are not finitely generated, hence not invertible [5, Corollary V.2.4]).

For the maximal ideals, we could also use the easy fact that Int(D) is completely integrally closed if and only if the same holds for D [5, Exercise VI.10], and then note that, for a completely integrally closed domain, we have the same conclusion as (a) above [13, Corollary 34.4].

We end this section with two examples. The first is a nondivisorial ideal $\mathfrak{A}$ with $\Psi_{\mathfrak{A}^{-1}} = -\Psi_{\mathfrak{A}}$.

Example 3.7. The ideal $\mathfrak{A} = (1/X)\mathfrak{P}_X$ is not divisorial (since $\mathfrak{P}_X$ is not divisorial), and yet $\Psi_{\mathfrak{A}^{-1}} = -\Psi_{\mathfrak{A}}$. Indeed, $\mathfrak{A}^{-1} = X\mathfrak{P}_X^{-1} = X\,\mathrm{Int}(V)$ [Proposition 3.5]. If $x \neq 0$, then $\Psi_{\mathfrak{A}}(x) = -v(x) + \Psi_{\mathfrak{P}_X}(x) = -v(x)$, and $\Psi_{\mathfrak{A}^{-1}}(x) = v(x)$. If $x=0$, then $\Psi_{\mathfrak{A}^{-1}}(0) = \infty$; we must show that $\Psi_{\mathfrak{A}}(0) = -\infty$. In case V is the valuation ring of the p-adic valuation in $\mathbb{Q}$, let $h_n = (X-1)\cdots(X-n+1)/n!$, then $h_n \in \mathfrak{A}$ and $h_n(0) = (-1)^n \frac{1}{n}$. The general case may be handled in a similar way by using an appropriate regular basis of Int(V); see [5, Section II.2].

The second example is a divisorial ideal which is not invertible, as in [5, Example VIII.5.16]. We use it below for the global case [Example 4.3].

Example 3.8. As in Example 2.5, we consider the clopen sets $U_n = t^n + \widehat{\mathfrak{m}}^{n+2}$. We then let Ψ_n be the locally constant function with value 1 on U_n and 0 outside. It is the value-function of a finitely generated (thus invertible) unitary ideal $\mathfrak{A}_n$ [Proposition 1.6]. The intersection $\mathfrak{A} = \bigcap_{n=0}^{\infty} \mathfrak{A}_n$ is a divisorial ideal since it is the intersection of invertible ideals. The nonzero constant t is in each $\mathfrak{A}_n$ (since $v(t) \geq \Psi_n(x)$, for each $x \in \widehat{V}$) so that $\mathfrak{A}$ is a unitary ideal. To show that $\mathfrak{A}$ is not finitely generated, it suffices to prove that $\Psi_{\mathfrak{A}}$ is not locally constant. We prove that, for each k, we have $\Psi_{\mathfrak{A}}(t^k) \geq 1$ and $\Psi_{\mathfrak{A}}(t^k + t^{k+1}) = 0$. Indeed, $\Psi_{\mathfrak{A}}(t^k) \geq 1$, since $\mathfrak{A} \subseteq \mathfrak{A}_k$. On the other hand, the clopen set $W_k = t^k + t^{k+1} + \widehat{\mathfrak{m}}^{k+2}$ does not meet any of the clopen sets U_n. From the Stone–Weierstrass theorem, there is a polynomial g_k whose value-function Ψ_{g_k} takes the value 0 on W_k and 1 outside. Hence $\Psi_{g_k} \geq \Psi_n$, and it follows from Theorem 1.3 that we have $g_k \in \mathfrak{A}_n$. Since this holds for each n, we have $g_k \in \mathfrak{A}$, hence $\Psi_{\mathfrak{A}} \leq \Psi_{g_k}$. Therefore $\Psi_{\mathfrak{A}}(t^k + t^{k+1}) = 0$.

Finally, we note that $\Psi_{\mathfrak{A}^{-1}} \neq -\Psi_{\mathfrak{A}}$, even though $\mathfrak{A}$ is divisorial. Indeed, it follows from Lemma 3.1 that $\Psi_{\mathfrak{A}^{-1}}(t^k + t^{k+1}) \geq -\Psi_{\mathfrak{A}}(t^k + t^{k+1}) = 0$, whence, from Lemma 2.2, we have $\Psi_{\mathfrak{A}^{-1}}(0) \geq 0$. However, again by Lemma 2.2, $\Psi_{\mathfrak{A}}(0) \geq 1$, since $\Psi_f(t^k) \geq 1$.

4. Globalization

We now let D be a domain with quotient field K and are mainly concerned with the case where D is a Dedekind domain with finite residue fields. For each maximal ideal $\mathfrak{m}$ of D, $D_{\mathfrak{m}}$ is then a discrete valuation domain; we denote by $v_{\mathfrak{m}}$ the corresponding valuation and by $\widehat{D_{\mathfrak{m}}}$ the completion of $D_{\mathfrak{m}}$ with respect to $v_{\mathfrak{m}}$. We define the $\mathfrak{m}$-adic value-function $\Psi_{\mathfrak{m},\mathfrak{A}}$ of a fractional ideal $\mathfrak{A}$ of $\mathrm{Int}(D)$ as

$$\Psi_{\mathfrak{m},\mathfrak{A}}(x) = \inf\{v_{\mathfrak{m}}(\varphi(x)) \mid \varphi \in \mathfrak{A}\}.$$

This is a function from $\widehat{D_{\mathfrak{m}}}$ to $\mathbb{Z} \cup \{-\infty, \infty\}$. Recall that, for each maximal ideal $\mathfrak{m}$ of D, we have $(\mathrm{Int}(D))_{\mathfrak{m}} = \mathrm{Int}(D_{\mathfrak{m}})$ [4, Corollaire 5, p. 303]. Hence, for each fractional ideal $\mathfrak{A}$ of $\mathrm{Int}(D)$, the function $\Psi_{\mathfrak{m},\mathfrak{A}}$ is the value-function of the ideal $\mathfrak{A}_{\mathfrak{m}}$ of $\mathrm{Int}(D_{\mathfrak{m}})$. As in the local case, let us say that a fractional ideal of $\mathrm{Int}(D)$ is *almost-unitary* if it is the product of a unitary ideal by a nonzero constant. We derive immediately the following from Theorem 1.3.

Theorem 4.1. *Let D be a Dedekind domain with finite residue fields and $\mathfrak{A}$, $\mathfrak{B}$ be two almost-unitary (fractional) ideals of* $\mathrm{Int}(D)$. *Then $\mathfrak{A} = \mathfrak{B}$ if and only if, for each maximal ideal $\mathfrak{m}$ of D, we have $\Psi_{\mathfrak{m},\mathfrak{A}} = \Psi_{\mathfrak{m},\mathfrak{B}}$.*

The following result generalizes Corollary 1.7.

Proposition 4.2. *Let D be a Dedekind domain with finite residue fields and $\mathfrak{A}$ be an almost-unitary (fractional) ideal of* $\mathrm{Int}(D)$. *Then $\mathfrak{A}$ is finitely generated if and only if, for each maximal ideal $\mathfrak{m}$ of D, the $\mathfrak{m}$-adic value-function $\Psi_{\mathfrak{m},\mathfrak{A}}$ is locally constant.*

Proof. If $\mathfrak{A}$ is finitely generated, then so is each $\mathfrak{A}_{\mathfrak{m}}$, and each $\mathfrak{m}$-adic value-function $\Psi_{\mathfrak{m},\mathfrak{A}}$ is locally constant. For the converse, we may as well assume that $\mathfrak{A}$ is a unitary ideal (since $a\mathfrak{A}$ is finitely generated if and only if $\mathfrak{A}$ is finitely generated), hence that $\mathfrak{A}$ is contained in $\mathrm{Int}(D)$ and contains a nonzero constant d. There are finitely many maximal ideals $\mathfrak{m}$ containing d. For each such $\mathfrak{m}$, $\mathfrak{A}_{\mathfrak{m}}$ is finitely generated [Corollary 1.7] and we may choose a set of generators contained in $\mathfrak{A}$. Let $\mathfrak{B}$ be the ideal (finitely) generated by d and the finite union of these finite sets of generators. Then $\mathfrak{A}=\mathfrak{B}$ since, for each maximal ideal $\mathfrak{m}$ of D, we have $\mathfrak{B}_{\mathfrak{m}}=\mathfrak{A}_{\mathfrak{m}}$ (this is clear if $\mathfrak{m}$ contains d, from the definition of $\mathfrak{B}$, and in the other case, d is a unit in $D_{\mathfrak{m}}$, and thus $\mathfrak{B}_{\mathfrak{m}}=\mathfrak{A}_{\mathfrak{m}}=\mathrm{Int}(D_{\mathfrak{m}})$). □

For every Dedekind domain D with finite residue fields we can produce an example of a divisorial ideal of Int(D) which is not invertible, as in the local case [Example 3.8]. We note that Alan Loper has also given such an example (in Int($\mathbb{Z}$)) [16].

Example 4.3. Choose a maximal ideal $\mathfrak{m}$ and a generator t of $\mathfrak{m}D_{\mathfrak{m}}$ in D. In Example 3.8 we considered a sequence of unitary finitely generated ideals $\mathfrak{A}_n$ of $\mathrm{Int}(D_{\mathfrak{m}})$, each one corresponding to a locally constant function Ψ_n (on $\widehat{D_{\mathfrak{m}}}$) such that Ψ_n takes only the values 0 or 1, $\Psi_n(t^n)=1$, and $\Psi_n(t^k+t^{k+1})=0$ for each k and each n. Let $\mathfrak{B}_n=(\mathfrak{A}_n\cap \mathrm{Int}(D))+\mathfrak{m}\,\mathrm{Int}(D)$. Then $\mathfrak{B}_n$ is a unitary finitely generated ideal of Int(D). If $\Psi_{\mathfrak{m},\mathfrak{B}_n}$ denotes the $\mathfrak{m}$-adic value-function of $\mathfrak{B}_n$ then $\Psi_{\mathfrak{m},\mathfrak{B}_n}=\Psi_n$. On the other hand, note that for each maximal ideal $\mathfrak{n}\neq\mathfrak{m}$, the $\mathfrak{n}$-adic value-function of $\mathfrak{B}_n$ is null (since $\mathfrak{B}_n$ contains $\mathfrak{m}$). The intersection $\mathfrak{B}=\bigcap_{n=0}^{\infty}\mathfrak{B}_n$ is a divisorial ideal since it is the intersection of invertible ideals. It contains $\mathfrak{m}$, hence it is a unitary ideal. Finally we show, as in Example 3.8, that the $\mathfrak{m}$-adic value-function $\Psi_{\mathfrak{m},\mathfrak{B}}$ of $\mathfrak{B}$ is not locally constant, and it follows from Proposition 4.2 that $\mathfrak{B}$ is not finitely generated. On the one hand, for each k, we have $\Psi_{\mathfrak{m},\mathfrak{B}}(t^k)\geq 1$, since $\mathfrak{B}\subseteq\mathfrak{B}_k$. On the other hand, we have seen in Example 3.8 that there is a polynomial g_k (that we may choose in Int(D)) such that $v_{\mathfrak{m}}(g_k(x))\geq\Psi_n(x)$, for each $x\in\widehat{D_{\mathfrak{m}}}$ and each n, and also such that $v_{\mathfrak{m}}(g_k(t^k+t^{k+1}))=0$. Since the $\mathfrak{n}$-adic value-function of $\mathfrak{B}_n$ is null, for $\mathfrak{n}\neq\mathfrak{m}$, it follows from Theorem 4.1 that $g_k\in\mathfrak{B}_n$. Finally $g_k\in\mathfrak{B}$, hence $\Psi_{\mathfrak{m},\mathfrak{B}}(t^k+t^{k+1})\leq v_{\mathfrak{m}}(g_k(t^k+t^{k+1}))=0$.

We determine now which prime ideals are invertible or divisorial. We shall see that, as in the local case, the two conditions are equivalent [Proposition 3.5], but also that, contrary to this case, an upper to zero which is maximal is not necessarily invertible. We then start with some considerations on uppers to zero, relaxing first the hypothesis that D is a Dedekind domain. Recall that the uppers to zero of Int(D) are always of the form $\mathfrak{P}_q=qK[X]\cap\mathrm{Int}(D)$, where q is an irreducible polynomial in $K[X]$ (that can be chosen with coefficients in D) [5, Corollary V.1.2].

Lemma 4.4. *Let D be a domain (with quotient field K), and $q\in D[X]$ be a polynomial which is irreducible in $K[X]$. If the upper to zero $\mathfrak{P}_q$ is invertible, then there exists a nonzero constant $d\in D$ such that $(d/q)\in\mathfrak{P}_q^{-1}$.*

Proof. If $\mathfrak{P}_q\mathfrak{P}_q^{-1}=\mathrm{Int}(D)$, then $(\mathfrak{P}_qK[X])(\mathfrak{P}_q^{-1}K[X])=K[X]$. Since $\mathfrak{P}_qK[X]=qK[X]$, we derive $\mathfrak{P}_q^{-1}K[X]=(1/q)K[X]$, and the result follows immediately. □

When the upper to zero $\mathfrak{P}_q$ is maximal, we also have the following:

Lemma 4.5. *Let D be a domain (with quotient field K), $q \in D[X]$ be a polynomial irreducible in $K[X]$ such that the upper to zero $\mathfrak{P}_q$ is maximal, and d be a nonzero constant. Then the rational function d/q is such that $(d/q) \in \mathfrak{P}_q^{-1}$ if and only if it is integer-valued.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of D and $a \in D$. Since $\mathfrak{P}_q$ is maximal, it is not contained in the maximal ideal $\mathfrak{M}_{\mathfrak{m},a} = \{f \in \mathrm{Int}(D) \mid f(a) \in \mathfrak{m}\}$, hence there is $h \in \mathfrak{P}_q$ with $h(a) \notin \mathfrak{m}$. If $(d/q) \in \mathfrak{P}_q^{-1}$, then $(d/q)h$ is integer valued, and hence, $(d/q)(a) \in D_{\mathfrak{m}}$. This holds for each maximal ideal $\mathfrak{m}$ and each $a \in D$, hence it follows that d/q is integer valued.

Conversely, if the rational function d/q is integer valued, its product by an element of $\mathfrak{P}_q = qK[X] \cap \mathrm{Int}(D)$ is an integer valued polynomial, and hence, $(d/q) \in \mathfrak{P}_q^{-1}$. $\square$

Proposition 4.6. *Let D be a domain (with quotient field K), and $q \in D[X]$ be a polynomial irreducible in $K[X]$ such that the upper to zero $\mathfrak{P}_q$ is maximal. Then the following assertions are equivalent:*

(i) *$\mathfrak{P}_q$ is invertible,*
(ii) *there is a nonzero constant $d \in D$ such that $(d/q) \in \mathfrak{P}_q^{-1}$,*
(iii) *there is a nonzero constant $d \in D$ such that the rational function d/q is integer valued,*
(iv) *$\mathfrak{P}_q^{-1} \nsubseteq K[X]$,*
(v) *$\mathfrak{P}_q^{-1} \neq \mathrm{Int}(D)$,*
(vi) *$\mathfrak{P}_q$ is divisorial.*

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 4.4 and (ii) $\Leftrightarrow$ (iii) from Lemma 4.5.

(ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (iv): By way of contradiction, suppose that $\mathfrak{P}_q^{-1} \subseteq K[X]$ and let $\varphi \in \mathfrak{P}_q^{-1}$. Then there is a nonzero d in D such that $d\varphi \in \mathrm{Int}(D)$. Since $\mathfrak{P}_q$ is maximal and does not contain d, there are elements $h \in \mathfrak{P}_q$ and $f \in \mathrm{Int}(D)$ such that $1 = h + df$. Multiplying by φ, we obtain $\varphi = \varphi h + \varphi df \in \mathrm{Int}(D)$. Thus $\mathfrak{P}_q^{-1} = \mathrm{Int}(D)$.

(iv) $\Rightarrow$ (i): If $\mathfrak{P}_q$ is not invertible, then $\mathfrak{P}_q\mathfrak{P}_q^{-1} = \mathfrak{P}_q$. Hence $(\mathfrak{P}_qK[X])\mathfrak{P}_q^{-1} = \mathfrak{P}_qK[X]$, that is, $(qK[X])\mathfrak{P}_q^{-1} = qK[X]$. It follows that $\mathfrak{P}_q^{-1} \subseteq K[X]$.
Hence (i)–(v) are equivalent. On the other hand, (i) $\Rightarrow$ (vi) $\Rightarrow$ (v) is obvious, and hence, (vi) is also equivalent to (i). $\square$

Returning to the case where D is a Dedekind domain with finite residue fields, we show now that a prime ideal is invertible if and only it is divisorial. Recall that the unitary prime ideals above a maximal ideal $\mathfrak{m}$ of D are maximal and in one-to-one correspondence with the elements of $\widehat{D_{\mathfrak{m}}}$: to $\alpha \in \widehat{D_{\mathfrak{m}}}$, corresponds the maximal ideal

$$\mathfrak{M}_{\mathfrak{m},\alpha} = \{f \in \mathrm{Int}(D) \mid v_{\mathfrak{m}}(f(\alpha)) > 0\}.$$

Recall also that an upper to zero $\mathfrak{P}_q$ is contained in an $\mathfrak{M}_{\mathfrak{m},\alpha}$ if and only if $q(\alpha)=0$. (All this can immediately be derived from the local case [7].)

Proposition 4.7. *Let D be a Dedekind domain with finite residue fields and $\mathfrak{P}$ be a prime ideal of* $\operatorname{Int}(D)$.

(i) *$\mathfrak{P}$ is invertible if and only if it is divisorial. In fact, if $\mathfrak{P}$ is a nonzero prime ideal which is not invertible, then $\mathfrak{P}^{-1}=\operatorname{Int}(D)$.*

(ii) *If $\mathfrak{P}$ is invertible, then it is an upper to zero which is maximal.*

Proof. If $\mathfrak{P}$ is a maximal upper to zero which is not invertible, we have seen that $\mathfrak{P}^{-1}=\operatorname{Int}(D)$ [Proposition 4.6]. It remains to show that we have the same conclusion if $\mathfrak{P}$ is unitary, or an upper to zero which is not maximal.

- First consider the case where $\mathfrak{P}=\mathfrak{M}_{\mathfrak{m},\alpha}$ is a maximal unitary ideal. It follows from Proposition 3.5 that $(\mathfrak{P}_{\mathfrak{m}})^{-1}=\operatorname{Int}(D_{\mathfrak{m}})$. For each maximal ideal $\mathfrak{n}\neq\mathfrak{m}$, we have $\mathfrak{P}_{\mathfrak{n}}=\operatorname{Int}(D_{\mathfrak{n}})$, whence again $(\mathfrak{P}_{\mathfrak{n}})^{-1}=\operatorname{Int}(D_{\mathfrak{n}})$. Therefore $\mathfrak{P}^{-1}=\operatorname{Int}(D)$ (it is well known that, for each fractional ideal $\mathfrak{A}$ and each multiplicative subset S, we have $(\mathfrak{A}^{-1})_S\subseteq(\mathfrak{A}_S)^{-1}$).
- Next consider the case where $\mathfrak{P}=qK[X]\cap\operatorname{Int}(D)$ is an upper to zero which is not maximal. Let $\varphi\in\mathfrak{P}^{-1}$. For some maximal ideal $\mathfrak{m}$, $\mathfrak{P}_{\mathfrak{m}}=qK[X]\cap\operatorname{Int}(D_{\mathfrak{m}})$ is not maximal. For such an $\mathfrak{m}$, $\varphi\in(\mathfrak{P}_{\mathfrak{m}})^{-1}$, and it follows from Proposition 3.5 that $\varphi\in\operatorname{Int}(D_{\mathfrak{m}})$. In particular, $\varphi\in K[X]$ and, for each $a\in D$, we have $\varphi(a)\in D_{\mathfrak{m}}$. Now let $\mathfrak{n}$ be a maximal ideal for which $\mathfrak{P}_{\mathfrak{n}}$ is maximal, so that $\mathfrak{P}$ is not contained in any unitary prime ideal above $\mathfrak{n}$. For each $a\in D$, there is a polynomial $g\in\mathfrak{P}$ such that $g(a)\notin\mathfrak{n}$. It follows that φg is integer valued and that $\varphi(a)\in D_{\mathfrak{n}}$. Finally we may conclude that $\varphi\in\operatorname{Int}(D)$. $\square$

Contrary to the local case, we shall see that a maximal upper to zero is not always invertible when the characteristic of D is not zero [Example 4.11]. We first study under which condition no prime of $\operatorname{Int}(D)$ is invertible. This is linked with the Skolem property and the notion of d-ring simultaneously introduced by Brizolis [1] and Gunji and McQuillan [15]. So let us recall the following [5, Proposition VII.2.3; 15, Proposition 1]).

Proposition 4.8. *Let D be a domain (which is not a field). The following assertions are equivalent:*

(i) *each integer valued rational function on D is in fact a integer valued polynomial,*

(ii) *for each non constant polynomial f in $D[X]$, there exists an element $a\in D$ such that $f(a)$ is not a unit of D,*

(iii) *for each non constant polynomial f in $D[X]$, the intersection of the maximal ideals $\mathfrak{m}$ of D for which f has a root modulo $\mathfrak{m}$ is (0).*

Under these equivalent conditions, D is said to be a *d-ring*. It is known that if D is a Noetherian d-ring of characteristic zero, then no upper to zero in Int(D) is maximal [5, Lemma VII.5.11]. In fact, for a Dedekind domain with finite residue fields, whatever its characteristic, the d-ring property is equivalent to the fact that no (maximal upper to zero) prime ideal is invertible:

Proposition 4.9. *Let D be a Dedekind domain with finite residue fields. The following assertions are equivalent*:
(i) *D satisfies the Skolem property,*
(ii) *D is a d-ring,*
(iii) *no prime ideal of* Int(D) *is invertible.*

Proof. The equivalence of (i) and (ii) is recalled here for completeness. It holds even if D is a one-dimensional Noetherian domain with finite residue fields [5, Corollary VII.5.3].

Suppose now that some prime ideal of Int(D) is invertible. Such a prime is a maximal upper to zero $\mathfrak{P}_q$ [Proposition 4.7]. It follows from Proposition 4.6 that there is a nonzero constant d such that the rational function d/q is integer valued (but clearly not a polynomial). Therefore D is not a d-ring.

Conversely, if D is not a d-ring, some non-constant polynomial q in $D[X]$ takes only unit values on D. We may choose q to be irreducible in $K[X]$ [5, Exercise VII.6]. Clearly the upper to zero $\mathfrak{P}_q$ is principal (generated by q), thus a fortiori invertible. □

Although it follows from this proof that if there is an invertible prime ideal, then there is a principal prime ideal, we emphasize that, even in the local case, an invertible (upper to zero) prime ideal is not always principal (for a characterization of the principal prime ideals, see [5, Proposition VIII.5.6]).

It is known that the global rings of arithmetic, that is, rings of integers of an algebraic number field or a function field, are d-rings [5, Examples VII.2.12]. From Propositions 4.7 and 4.9 we then derive the following:

Corollary 4.10. *Let D be the ring of integers of an algebraic number field (resp., a function field), that is, the integral closure of* $\mathbb{Z}$ *(resp.,* $\mathbb{F}_q[T]$*) in a finite algebraic extension of* $\mathbb{Q}$ *(resp.,* $\mathbb{F}_q(T)$*). Then there are no divisorial prime ideals in* Int(D). *In fact, for each nonzero prime ideal* $\mathfrak{P}$ *of* Int(D), *we have* $\mathfrak{P}^{-1} = \mathrm{Int}(D)$.

We are ready for an example of a maximal upper to zero which is not invertible. (This is essentially the same as in [5, Exercise V.14] where we gave an example of a d-ring with a maximal upper to zero, without noting, however, that no prime of Int(D) was invertible.)

Example 4.11. Let $D = \mathbb{F}_q[T]$ be the ring of polynomials with coefficients in a finite field $\mathbb{F}_q$ of characteristic p, and let $g = X^p + T$. Then $\mathfrak{P}_g$ is a maximal upper to zero in $\mathrm{Int}(D)$ which is not invertible.

- Let us show that $\mathfrak{P}_g$ is maximal. Fix a maximal ideal $\mathfrak{m} = fD$ of D. It suffices to show that g has no root in the completion $\widehat{D_\mathfrak{m}}$. This completion is isomorphic to the ring of power series $\mathbb{F}_r[[Z]]$ with f corresponding to Z and $\mathbb{F}_q$ a subfield of $\mathbb{F}_r$. Writing T in $\mathbb{F}_r[[Z]]$ as $b(Z) = b_0 + b_1 Z + b_2 Z^2 + \cdots$, we claim that $b_1 \neq 0$. Indeed, if $f = f_0 + f_1 T + \cdots + f_n T^n$, then $Z = f_0 + f_1 b(Z) + \cdots + f_n b(Z)^n$ in $\mathbb{F}_r[[Z]]$. Thus $f_0 + f_1 b_0 + \cdots + f_n b_0^n = 0$ and $f_1 b_1 + 2 f_2 b_0 b_1 + \cdots + n f_n b_o^{n-1} b_1 = 1$. On the other hand, for each $\alpha \in \widehat{D_\mathfrak{m}}$, writing $\alpha = \sum c_i Z^i$ in $\mathbb{F}_r[[Z]]$, we obtain $\alpha^p = \sum c_i^p Z^{ip}$. Thus never $\alpha^p + T = 0$.
- It follows from Corollary 4.10 that $\mathfrak{P}_g$ is not invertible.

Another reason why $\mathfrak{P}_g$ is not invertible in the previous example, in connection with the fact that D is a d-ring, is that g has a (multiple) root modulo each maximal ideal $\mathfrak{m}$ of D. Assuming now that D is a Dedekind domain with finite residue fields, we may complete the characterization given in Proposition 4.6 as follows.

Lemma 4.12. *Let D be a Dedekind domain with finite residue fields, and $q \in D[X]$ be a polynomial which is irreducible in $K[X]$ such that the upper to zero $\mathfrak{P}_q$ is maximal. Then $\mathfrak{P}_q$ is invertible if and only if the set of maximal ideals $\mathfrak{m}$ of D such that q has a root modulo $\mathfrak{m}$ is finite.*

Proof. Suppose that $\mathfrak{P}_q$ is invertible. It follows from Proposition 4.6 that there is a nonzero constant $d \in D$ such that (d/q) is integer valued. If $\mathfrak{m}$ is a maximal ideal of D such that q has a root modulo $\mathfrak{m}$, that is, $q(a) \in \mathfrak{m}$, for some $a \in D$, then $d \in \mathfrak{m}$. Therefore the set of maximal ideals such that q has a root modulo $\mathfrak{m}$ is finite.

Conversely, for each maximal ideal $\mathfrak{m}$ of $\mathrm{Int}(D)$, the ideal $(\mathfrak{P}_q)_\mathfrak{m}$ is maximal, hence invertible in $\mathrm{Int}(D_\mathfrak{m})$ [Proposition 3.5]. It follows from Proposition 4.6 that there is a nonzero constant $d \in D$ such that $(d/q) \in \mathrm{Int}(D_\mathfrak{m})$. If the set of maximal ideals such that q has a root modulo $\mathfrak{m}$ is finite, let $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ be these ideals and $d_1, \ldots, d_s$ be the corresponding nonzero constants. It follows that the rational function $(\prod_{i=1}^s d_i)/q$ is integer valued. From Proposition 4.6 again, we may conclude that $\mathfrak{P}_q$ is invertible. $\square$

If D is of characteristic 0, we can derive that the maximal uppers to zero are invertible from the previous lemma and the following.

Lemma 4.13. *Let D be a Dedekind domain with finite residue fields and $q \in D[X]$ be a polynomial which is irreducible in $K[X]$ such that the upper to zero $\mathfrak{P}_q$ is maximal. If D is of characteristic 0, then the set of maximal ideals $\mathfrak{m}$ of D such that q has a root modulo $\mathfrak{m}$ is finite.*

Proof. For each maximal ideal $\mathfrak{m}$ of D, q has no root in $\widehat{D_{\mathfrak{m}}}$, since $\mathfrak{P}_q$ is maximal. Suppose that $\mathfrak{m}$ is a maximal ideal of D such that q has a root modulo $\mathfrak{m}$. It follows from Hensel's lemma [19, Theorem (44.4)] that such a root of q must be a multiple root, hence also a root of the derivative q'. Since q is irreducible and the characteristic of D is 0, q and q' are coprime in $K[X]$. Hence there are polynomials u and v, with coefficients in D, and a nonzero constant d such that $uq + vq' = d$. Then $\mathfrak{m}$ contains d. Finally there are finitely many such maximal ideals $\mathfrak{m}$. □

As in the local case, we thus obtain a complete characterization:

Theorem 4.14. *Let D be a Dedekind domain with finite residue fields and $\mathfrak{P}$ be a prime ideal of* Int(D). *If D is of characteristic* 0, *the following assertions are equivalent*:

(i) *the ideal $\mathfrak{P}$ is invertible,*

(ii) *the ideal $\mathfrak{P}$ is divisorial,*

(iii) *the ideal $\mathfrak{P}$ is an upper to zero which is maximal.*

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